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# Homotopy perturbation Laplace transform solution of fractional non-linear reaction diffusion system of Lotka-Volterra type differential equation

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ABSTRACT

This work investigates the fractional non linear reaction diffusion (FNRD) system of Lotka-Volterra type. The system of equations together with the boundary conditions are solved by Homotopy perturbation transform method (HPTM). The series solutions are obtained for the two cases (homogeneous and non-homogeneous) of FNRD system. The effect of fractional parameter on the mass concentration of two species are shown and discussed with the help of 3D graphs.



## 1. Introduction

Fractional differential equations are widely used to model the fluid flow problems. The application of fractional derivatives and fractional curl operators is growing because of its wide utility in mathematical biology [1–3,7], wave motion [4], coupled oscillations in engineering [5,6], physics [8] mathematical finance [9] and in various branches of modern science and technology. More recently some important and further applications of fractional calculus has also found its applications in earth quake oscillations [10,11] and in electrical impedances in botanical elements i.e., fruits and vegetables [12,13].

The continued interest in reaction-diffusion systems stem from the fact that such systems with variable diffusitivities generalize many non-linear models which are widely used in modern day scientific and research problems. As a particular case reaction-diffusion system corresponds to one population diffusion system for two species in terms of Lotka-Volterra type [14,15]. In this paper we consider the fractional non-linear reaction-diffusion system with variable diffusitivities as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(D_1(u)\frac{\partial u}{\partial x}\right) + F(u, v)$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(D_2(u)\frac{\partial v}{\partial x}\right) + G(u, v), \quad 0 < \alpha \leq 1.$$

Recently fractional order model has received considerable attention in mathematical biology. Ozalp [16] studied fractional order SEIR model using transverse transmission. Fractional differential equation for HIV model is analyzed by Kou et al. [17]. Malmivuo and Plonsey [18] computed fractional dynamical models arising from biological systems. It is difficult to find exact solution of fractional non-linear differential equations. Several analytical and numerical techniques are available in literature [19–27] to solve such differential equations. Homotopy perturbation method has been employed to solve non-linear fractional differential equations. Another approach to solve such equations is by the combination of various methods with Laplace transform method e.g., Adomian decomposition transform method (ADTM) [28], Differential transform method (DTM) [29], Homotopy analysis transform method (HATM) [30] and Homotopy perturbation transform method (HPTM) [31].

In this paper, we discussed the implementation of (HPTM) to solve the fractional non-linear reaction diffusion (FNRD) system of Lotka-Volterra type. The analytic solution in series form is obtained by HPTM. The graphical results are also obtained for different values of fractional parameter. The correctness of the obtained results is verified when fractional order derivatives are replaced by ordinary derivatives.

## 2. Homotopy perturbation transform method (HPTM)

**Definition 1.1.** The fractional derivative of $u(t)$ is defined as [8]

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$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}u(t)d\xi, \quad 0<\alpha<1,$$

**Definition 1.2.** The Laplace transform of function $f(t)$ is defined as [32]

$$F(s) = L[f(t)] = \int_0^\infty e^{-st}f(t)dt. \tag{1}$$

**Definition 1.3.** The Laplace transform of the fractional derivative is defined as [32]

$$L\{D^{n\alpha}f(t)\} = s^{n\alpha}F(s) - \sum_{k=0}^{n-1} s^{n\alpha-k-1}f^{(k)}(0), \tag{2}$$

where $n-1<\alpha\le n$.

*2.1. Methodology of HPTM for non-linear fractional differential equations*

In order to elucidate the solution procedure of the fractional Laplace Homotopy perturbation transform method, we construct the homotopic structure [33] for the following problem

$$D_t^\alpha w(x,t) = Rw(x,t) + Nw(x,t) + r(x,t), \quad t>0, 0<\alpha <1, w(x,0) = w_0(x). \tag{3}$$

The homotopic structure is

$$H(w,p) = (1-p)w(x,t) + p(Lw(x,t) + Nw(x,t) + r(x,t)) = 0.$$

The embedding parameter $p$ monotonically increases from 0 to 1.

$$(1-p)\mathcal{L}[D_t^a w(x,t)] + p\mathcal{L}[D_t^a w(x,t) - Rw(x,t) + Nw(x,t) + r(x,t)] = 0,$$

where $R$ is the linear and $N$ is the non-linear operator in the variable $x, r(x,t)$ is continuous function,$w(x,0)$ is the initial condition. The first step towards the solution of the mentioned method is applying Laplace transform on both sides of the Eq. (3), we have

$$L\{D_t^\alpha w(x,t)\} = L\{Rw(x,t) + Nw(x,t) + r(x,t)\}. \tag{4}$$

On applying the derivative rule of the Laplace transform on equation Eq. 4, we get

$$L\{w(x,t)\} = s^{-1}w_0(x) + s^{-\alpha}L\{Rw(x,t) + Nw(x,t) + r(x,t)\}. \tag{5}$$

Here we can assume that the solution of the given equation can be expressed as power series in the following form

$$w(x,t) = \sum_{n=0}^\infty p^n w_n(x,t), \tag{6}$$

and when $p\to 0$ zeroth order approximation $w_0(x,t)$ is obtained and $p\to 1$ gives the final solution $w(x,t)$. The non-linear term can be expressed as

$$N\{w(x,t)\} = \sum_{n=0}^\infty p^n H_n(w), \tag{7}$$

here $H_n$ are He's polynomials and can be evaluate by using the formula given below [33]

$$H_n(w_1,w_2,w_3,\ldots,w_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}N\left(\sum_{j=0}^\infty p^j w_i\right)|_{p=0}.$$

On substituting He's polynomial in Eq. 8 and the resulting expression in Eq. 5, we obtain

$$L\left[\sum_{n=0}^\infty p^n w_n(x,t)\right] = (s^{-1}w_0(x) + \bar{r}(x,s)) + p\left[s^{-\alpha}L\left\{R\sum_{n=0}^\infty p^n w_n + \sum_{n=0}^\infty p^n H_n(w_0,w_1,\ldots\ldots w_n)\right\}\right]. \tag{8}$$

Now applying inverse Laplace transform to Eq. 9, we have

$$\sum_{n=0}^\infty p^n w_n(x,t) = F(x,t) + p\left[L^{-1}\left\{s^{-\alpha}L\left\{R\sum_{n=0}^\infty p^n w_n + \sum_{n=0}^\infty p^n H_n\right\}\right\}\right], \tag{9}$$

where $F(x,t)$ is the term obtained by the initial condition and source term. From Eq. 10, we obtain the following approximations as

$$p^0: w_0(x,t) = F(x,t), \tag{10}$$

$$p^1: \quad w_1(x,t) = L^{-1}[s^{-\alpha}L\{Rw_0 + H_0(w_0)\}], \tag{11}$$

$$p^2: \quad w_2(x,t) = L^{-1}[s^{-\alpha}L\{Rw_1 + H_1(w_0,w_1)\}], \tag{12}$$

$$p^3: \quad w_3(x,t) = L^{-1}[s^{-\alpha}L\{Rw_2(x,t) + H_2(w)\}], \tag{13}$$

and

$$p^n: \quad w_n(x,t) = L^{-1}[s^{-\alpha}L\{Rw_{n-1}(x,t) + H_{n-1}(w_0,w_1,\ldots\ldots.w_{n-1})\}]. \tag{14}$$

Finally we obtain the semi-analytic solution in the form of truncated series of approximation as

$$w(x,t) = \lim_{p\to 1}\left(\sum_{n=0}^\infty p^n w_n(x,t)\right). \tag{15}$$

## 3. Illustrative example

In this section we shall demonstrate the application of HPTM to solve FNRD system of Lotka-Volterra type differential equation.

$$D_t^\alpha u = (uu_x)_x + u(a_1 + b_1 u) + c_1 v + d_1, \tag{16}$$

$$D_t^\alpha v = (vv_x)_x + v(a_2 + b_2 v) + c_2 u + d_2, \tag{17}$$

where $a_i, b_i, c_i$ and $d_i$ $i=1,2$ are arbitrary constants with $b_1b_2\neq 0, c_1c_2\neq 0, b_1=b_2=b>0, d_1=\frac{2c_1a_1-6c_1^2}{b}$, $d_2 = d_1 + \frac{4c_1}{b}(3c_1-a_1)$, $a_2 = a_1 - 6c_1, c_2 = -c_1$, with initial conditions

$$u(x,0) = \phi_0 \pm \left(\phi_0 + \frac{2c_1}{b}\right)\cos\left[\sqrt{\frac{b}{2}}x - \beta_0\right] = f(x), \tag{18}$$

$$v(x,0) = \phi_0 + \frac{4c_1}{b} \pm \frac{|c_1|}{c_1}\left(\phi_0 + \frac{2c_1}{b}\right)\sin\left[\sqrt{\frac{b}{2}}x - \beta_0\right] = g(x), \tag{19}$$

with

$$\phi_0 = \frac{1}{3b}\begin{cases} \frac{2}{t_0} - 6c_1 & a_1 = 3c_1, \\ |3c_1 - a_1|\tanh\left(\frac{|3c_1-a_1|t_0}{2}\right) & a_1\neq 3c_1. \end{cases} \tag{20}$$

Taking Laplace transform of Eqs. 17 & 18 we have

$$s^\alpha\bar{u}(x,s) - u(x,0) = \frac{d_1}{s} + L\{a_1u + c_1v + b_1u^2 + (uu_x)_x\}, \tag{21}$$

$$(\bar{u}(x,s)) = \frac{f(x)}{s^\alpha} + \frac{d_1}{s^{\alpha+1}} + \frac{L}{s^\alpha}\{a_1u + c_1v + b_1u^2 + (uu_x)_x\}, \tag{22}$$

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Taking inverse Laplace transform and applying HPM, Eqs. 22 & 23 yield

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t)\text{and}\,v(x,t)=\sum_{n=0}^{\infty}p^n v_n(x,t),$$

$$\sum n=0^{\infty}p^n u_n=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}f(x)+\frac{t^{\alpha}}{\Gamma(\alpha)}d_1 +pL^{-1}\left[\frac{1}{s^{\alpha}}L\left\{a_1\sum_{n=0}^{\infty}p^n u_n+c_1\sum_{n=0}^{\infty}p^n u_n\sum_{n=0}^{\infty}p^n H_n(u_0,u_1,u_2,\ldots,u_n)\right\}\right], \tag{23}$$

and

$$\sum_{n=0}^{\infty}p^n v_n=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}f(x)+\frac{t^{\alpha}}{\Gamma(\alpha)}d_1 +pL^{-1}\left[\frac{1}{s^{\alpha}}L\left\{a_2\sum_{n=0}^{\infty}p^n v_n+c_2\sum_{n=0}^{\infty}p^n u_n\sum_{n=0}^{\infty}p^n H_n(v_0,v_1,v_2,\ldots,v_n)\right\}\right]. \tag{24}$$

In order to obtain the solution we shall compare different powers of $p^0$, $p^1, p^2$ which yield

$$p^0\ :\begin{cases} u_0=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}f(x)+\frac{t^{\alpha}}{\Gamma(\alpha)}d_1,\\ v_0=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}g(x)+\frac{t^{\alpha}}{\Gamma(\alpha)}d_2,\end{cases} \tag{25}$$

$$p^1\ :\begin{cases} u_1=L^{-1}\left[\frac{1}{s^{\alpha}}L\{a_1u_0+c_1v_0+H_0(u_0)\}\right],\\ v_1=L^{-1}\left[\frac{1}{s^{\alpha}}L\{a_1v_0+c_1u_0+H_0(v_0)\}\right],\end{cases} \tag{26}$$

and

$$p^n:\begin{cases} u_n=L^{-1}\left[s^{-\alpha}L\left\{\begin{array}{c} a_1u_{n-1}+c_1v_{n-1}+\\ \sum_{m=0}^{n-1}b_1u_{n-1-m}u_m+u'_{n-1-m}u'_m+u_{n-1-m}u''_m\end{array}\right\}\right],\\ v_n=L^{-1}\left[s^{-\alpha}L\left\{\begin{array}{c} a_2v_{n-1}+c_2u_{n-1}\\ \sum_{m=0}^{n-1}b_2v_{n-1-m}v_m+v'_{n-1-m}v'_m+v_{n-1-m}v''_m\end{array}\right\}\right].\end{cases} \tag{27}$$

**Case:1. Homogeneous Fractional non-linear diffusion system of Lotka-Volterra type**

For $a_1=3c_1,\phi_0=\frac{1}{3b}\left(\frac{2}{t_0}-6\right)$, $c_1=1,b_1=b_2=b,c_2=-1,a_2=a_1-6c_1,d_1=d_2=0$, the zeroth order solution of Lotka-Volterra type system will take the form

$$u_0=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\left[\frac{1}{3b}\left(\frac{2}{t_0}-6\right)+\left(\frac{1}{3b}\left(\frac{2}{t_0}-6\right)+\frac{2}{b}\right)\cos\left(\sqrt{\frac{b}{2}}x-\beta_0\right)\right] \tag{28}$$

and

$$v_0=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\left[\frac{1}{3b}\left(\frac{2}{t_0}-6\right)+\left(\frac{1}{3b}\left(\frac{2}{t_0}-6\right)+\frac{2}{b}\right)\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right)\right]. \tag{29}$$

$u_i(i=1,2,\ldots\ldots)$ and $v_i(i=1,2,\ldots..)$ can be obtained from Eq. 28.

**Case: 2. Non-homogeneous Fractional non-linear diffusion system of Lotka Volterra type**

For $a_1\neq 3c_1,a_1=3,d_1=1,c_1=1.2,b=3$, $d_2=1.96,c_2=-1.2,\phi_0=\frac{1}{3b}\left[\mid 3c_1-a_1\mid\tanh\left(\frac{|3c_1-a_1|t_0}{2}\right)-a_1-3c_1\right]$, the zeroth order is of the form

$$u_0=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\left[\phi_0+\left(\phi_0+\frac{2c_1}{b}\right)\cos\left(\sqrt{\frac{b}{2}}x-\beta_0\right)\right]+\frac{d_1t^{\alpha}}{\Gamma(\alpha)} \tag{30}$$

and

$$v_0=\frac{t^{\alpha-1}}{\Gamma(\alpha-1)}\left[\phi_0+\frac{4c_1}{b}+\left(\phi_0+\frac{2c_1}{b}\right)\sin\left(\sqrt{\frac{b}{2}}x-\beta_0\right)\right]+\frac{d_2t^{\alpha}}{\Gamma(\alpha)}, \tag{31}$$

and $u_i(i=1,2,\ldots\ldots)$ and $v_i(i=1,2,\ldots..)$ can be obtained from Eq. 28 and solution can be written as

$$u=\lim_{p\to 1}\sum_{n=0}^{\infty}p^n u_n,\qquad v=\lim_{p\to 1}\sum_{n=0}^{\infty}p^n v_n. \tag{32}$$

## 4. Numerical results and discussion

In this section, the graphical results of the solution for Eq. 31 are illustrated through Figs. 1–7 for various of $\alpha$. These figures show the behavior of an approximate solution truncated to the fifth term of the series obtained by HPTM for homogeneous and non homogeneous reaction diffusion system of equations when $\alpha=0.2,\alpha=0.7$, and $\alpha=0.9$. It is observed from Figs. 1–3 that mass concentration of both species $u(x,t)$ and $v(x,t)$ decreases with increases in $t$ for different values of $\alpha$. It shows that for the homogeneous case of non linear fractional reaction diffusion equation mass concentration of both species are decaying. Figs. 4–6 show the result for case 2. It is noted that mass concentration $u(x,t)$ is growing with the time $t$ while the mass concentration $v(x,t)$ decaying with time $t$ i.e., the non homogeneous term acts as a source to grow the concentration of first specie. Figs. 7,8 depict that magnitude of the mass concentration of both species are going to decrease by increasing the fractional parameter $\alpha$ for the homogeneous system of Lotka-Volterra type whereas for the non homogeneous system mass concentration of both species are decaying

**Table 1**
Residual error for the homogeneous fractional non-linear diffusion system of Lotka-Volterra types when $\alpha=0.9$.

| x | Residual error of u | Residual error of v |
|---|---|---|
| 0 | 0.0118679 | 0.00986739 |
| 0.1 | 0.0118506 | 0.00995241 |
| 0.2 | 0.0118238 | 0.0100348 |
| 0.3 | 0.0117879 | 0.0101137 |
| 0.4 | 0.0117431 | 0.0101883 |
| 0.5 | 0.01169 | 0.0102581 |
| 0.6 | 0.0116291 | 0.0103224 |
| 0.7 | 0.0115609 | 0.0103806 |
| 0.8 | 0.0114861 | 0.0104322 |
| 0.9 | 0.0114056 | 0.010477 |
| 1.0 | 0.0113201 | 0.0105144 |

**Table 2**
Residual error for the homogeneous fractional non-linear diffusion system of Lotkta-Volterra types when $\alpha=0.9$.

| x | Residual error of u | Residual error of v |
|---|---|---|
| 0.1 | 0.0580882 | 0.0251002 |
| 0.2 | 0.057807 | 0.0250156 |
| 0.3 | 0.0570113 | 0.0249473 |
| 0.4 | 0.0557133 | 0.0248966 |
| 0.5 | 0.0539323 | 0.0248643 |
| 0.6 | 0.051695 | 0.0248512 |
| 0.7 | 0.0490349 | 0.0248507 |
| 0.8 | 0.0459919 | 0.0248341 |
| 0.9 | 0.0426116 | 0.0247302 |
| 1.0 | 0.0389446 | 0.0247056 |

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**Case 1: For Homogeneous** (*FNRD*) **system** $(a_1 = 3c_1)$.

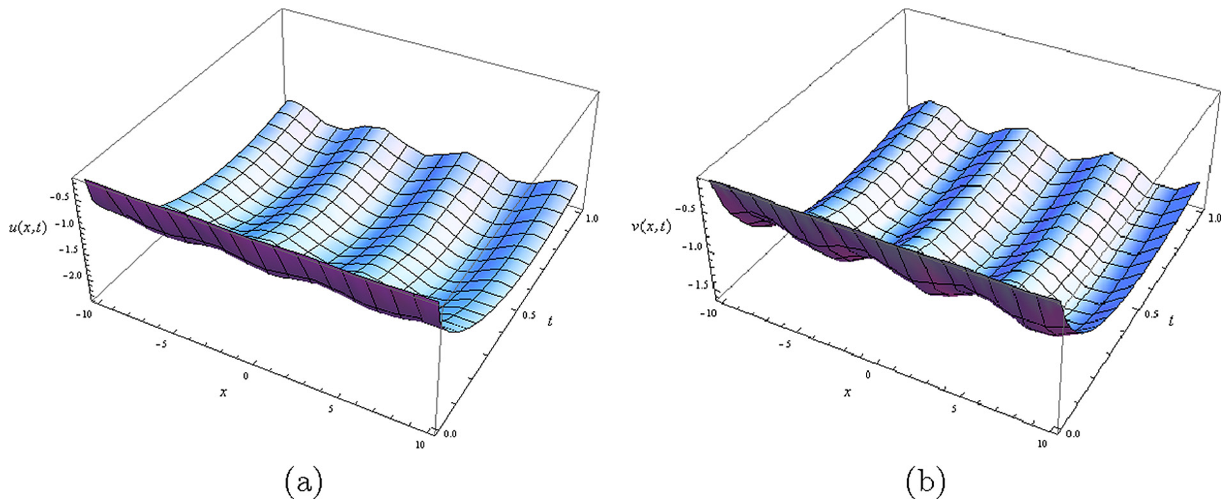


**Fig. 1.** Effect of fractional parameter on the mass concentration of two species u(x,t) and v(x,t) for $\alpha$ = 0.2.

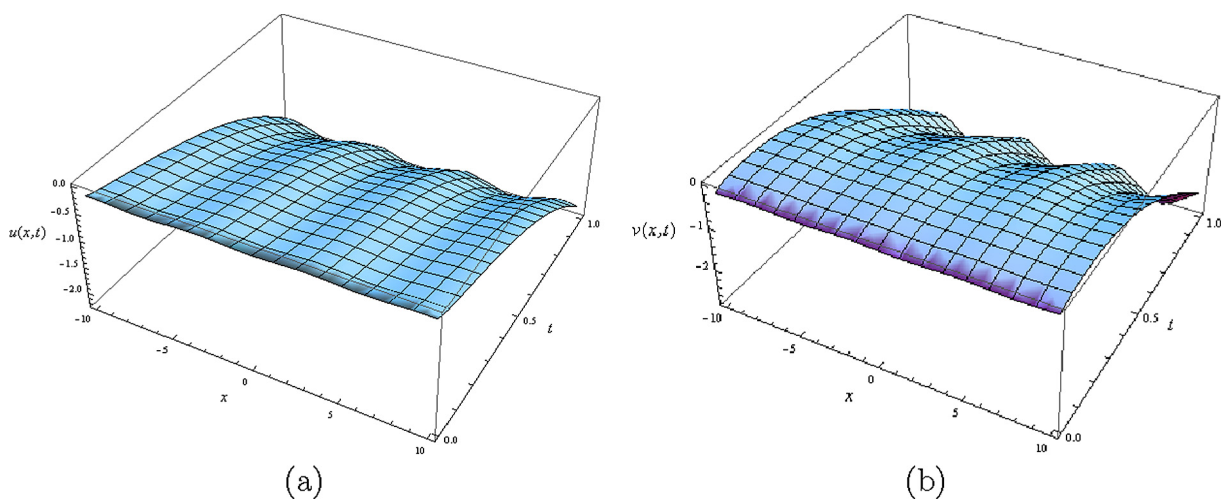


**Fig. 2.** Effect of fractional parameter on the mass concentration of two species $u(x,\ t)$ and $v(x,t)$ for $\alpha = 0.7$.

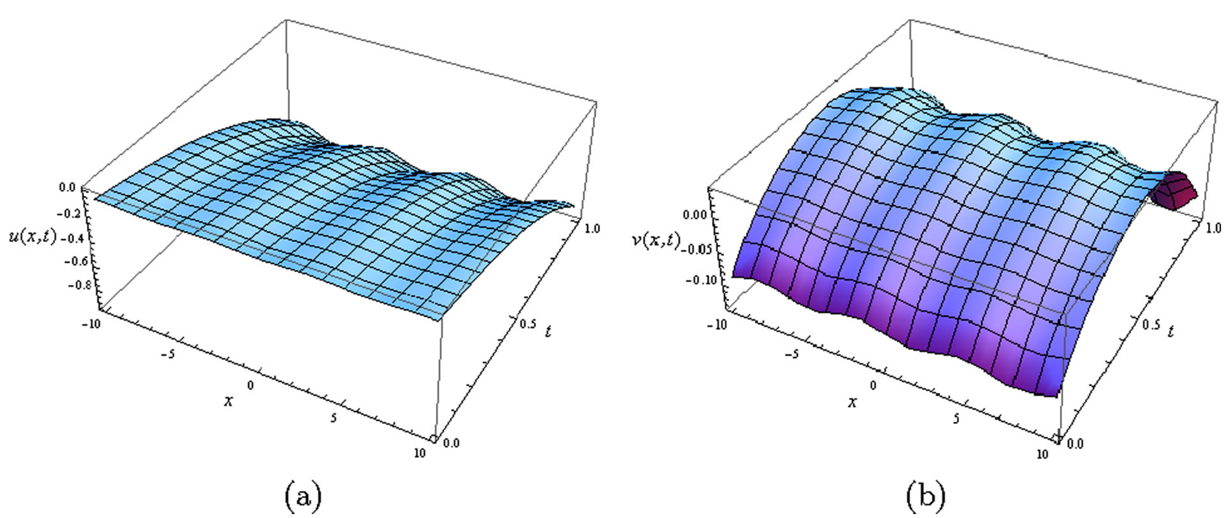


**Fig. 3.** Effect of fractional parameter on the mass concentration of two species $u(x,t)$ and $v(x,t)$ for $\alpha = 0.9$.

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**Case 2: For Non-homogeneous** (*FRND*) **system** $(a_1 \neq 3c_1)$.

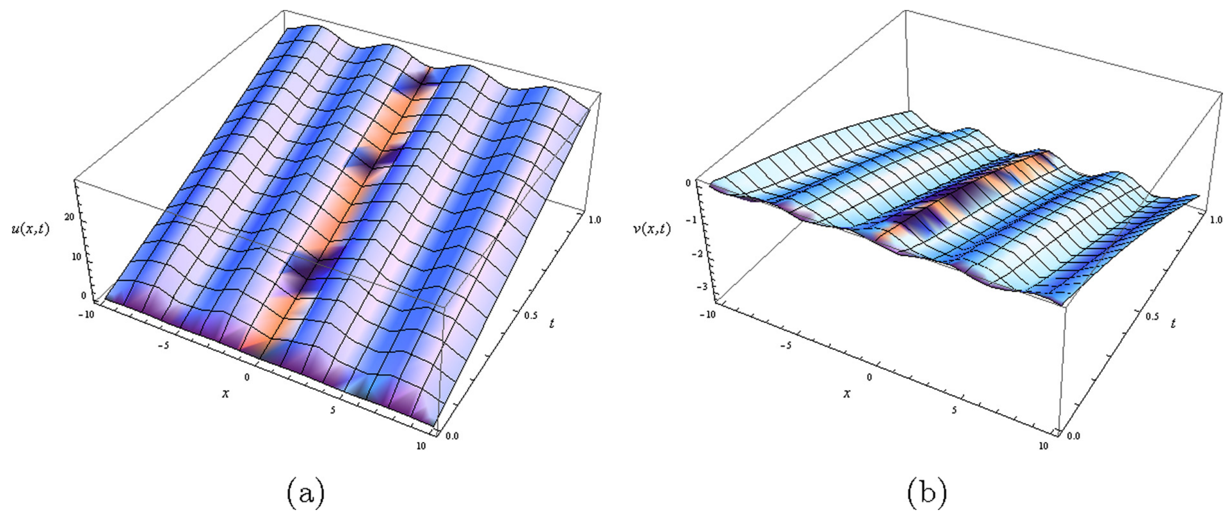


**Fig. 4.** Effect of fractional parameter on the mass concentration of two species $u(x,t)$ and $v(x,t)$ for $\alpha = 0.2$.

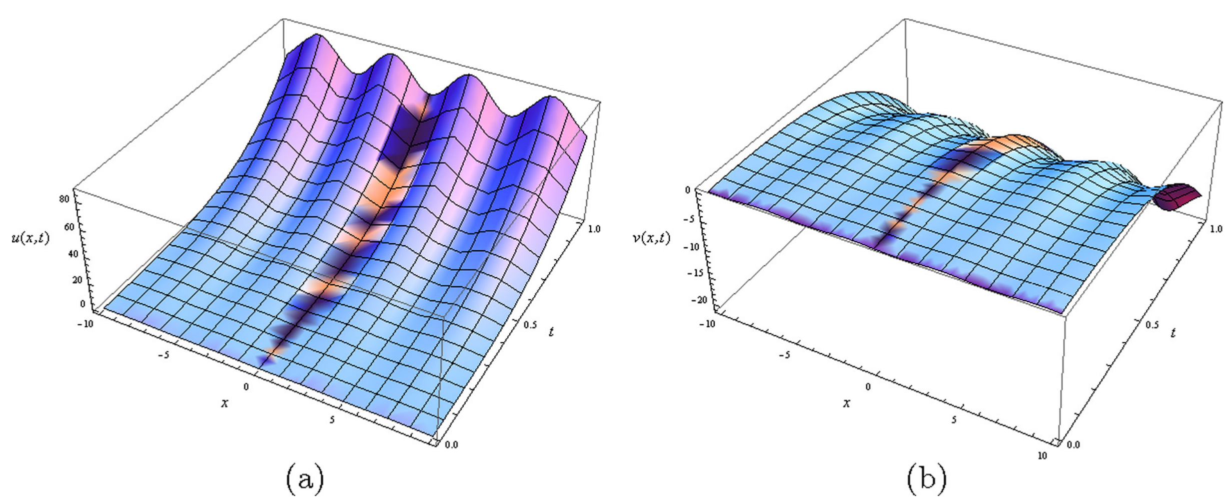


**Fig. 5.** Effect of fractional parameter on the mass concentration of two species $u(x,t)$ and $v(x,t)$ for $\alpha = 0.7$.

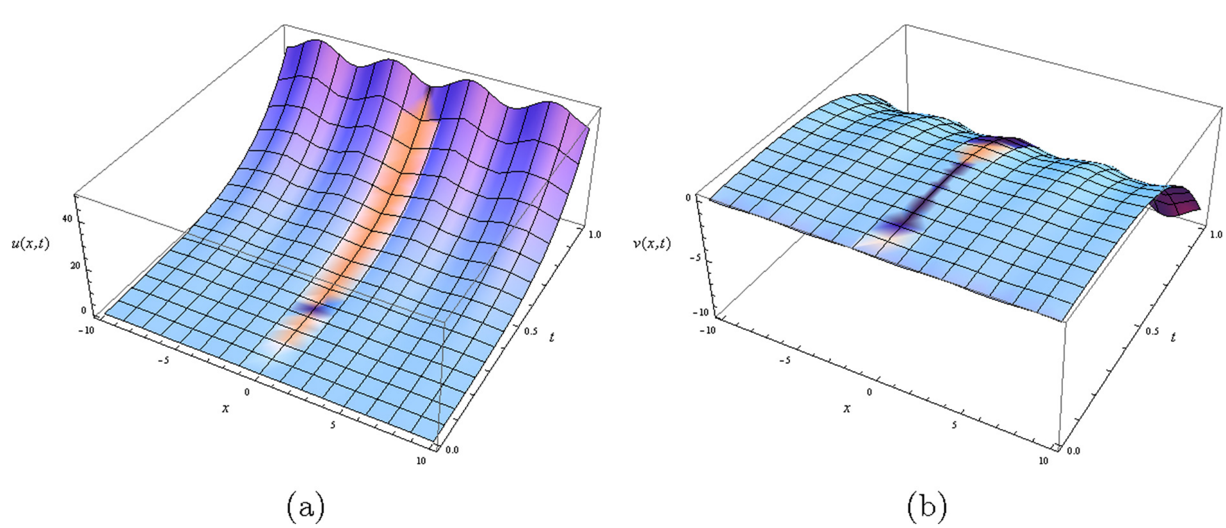


**Fig. 6.** Effect of fractional parameter on the mass concentration of two species $u(x,t)$ and $v(x,t)$ for $\alpha = 0.9$.

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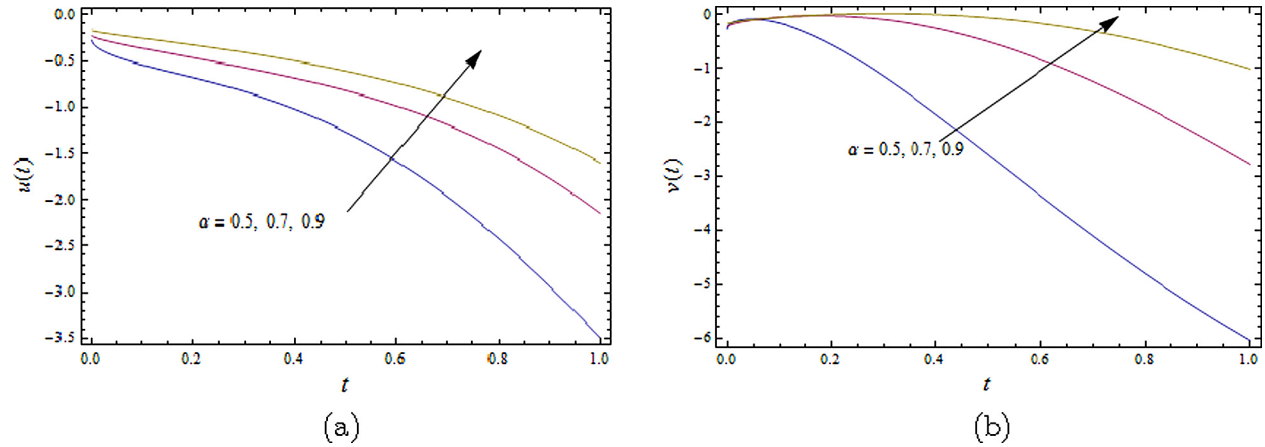


**Fig. 7.** Effect of fractional parameter on the mass concentration of two species $u(x,t)$ and $v(x,t)$ for $x = 1$. when $a_1 = 3c_1$.

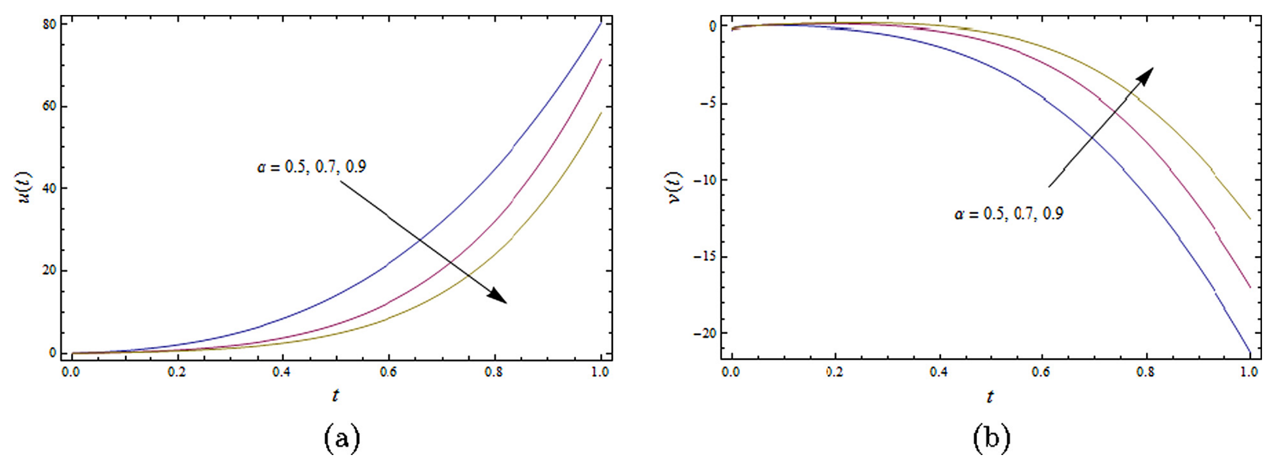


**Fig. 8.** Effect of fractional parameter on the mass concentration of two species $u(x,t)$ and $v(x,t)$ for $x = 1$. when $a_1 \neq 3c_1$.

by the fractional parameter $\alpha$. It can also be noted that the FNRD system reduces to the ordinary nonlinear fractional diffusion system when $\alpha = 1$ and our results match with the exact solution obtained by [15]. Table 1 and Table 2 show the residual errors of homogeneous and non-homogeneous fractional diffusion system of Lotka Volterra Type. It can be seen that our approximate solutions are convergent and the residual errors can be minimized by increasing the number of iterations.

## 5. Conclusions

- The mass concentration of both species decreases with increase in $t$ for different values of $\alpha$.
- The mass concentration $u(x,t)$ is growing with the time $t$ while the mass concentration $v(x,t)$ decaying with time $t$.
- The non homogeneous term acts as a source to grow the concentration of first specie.
- The mass concentration of both species are going to decrease by increasing the fractional parameter $\alpha$ for the homogeneous and non-homogeneous system of Lotka-Volterra type.
- The FNRD system reduces to the ordinary nonlinear fractional diffusion system when $\alpha = 1$ and our results match with the exact solution.

## References

[1] L. Sörnmo, P. Laguna, Bioelectrical Signal Processing in Cardiac and Neurological Applications, Academic Press vol. 1 (2005).
[2] S.R. Gois, M.A. Savi, An analysis of heart rhythm dynamics using a three coupled oscillator model, Chaos Solitons Fractals 41 (5) (2009) 2553–2565.
[3] A. Babloyantz, A. Destexhe, Is the normal heart a periodic oscillator, Biol. Cybern. 58 (3) (1988) 203–211.
[4] P.V.E. McClintock, A. Stefanovska, Noise and determinism in cardiovascular dynamics, Phys. A 314 (1) (2002) 69–76.
[5] A. Stefanovska, D.G. Luchinsky, P.V.E. McClintock, Modelling couplings among the oscillators of the cardiovascular system, Physiol. Meas. 22 (3) (2001) 551–564.
[6] A. Stefanovska, M. Bracic Lotric, S. Strle, H. Haken, The cardiovascular system as coupled oscillators?, Physiol Meas. 22 (3) (2001) 535–550.
[7] R. Eftimie, J.L. Bramson, D.J.D. Earn, Interactions between the immune system and cancer: a brief review of non-spatial mathematical models, Bull. Math. Biol. 73 (1) (2011) 2–32.
[8] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific vol. 12 (2000).
[9] Francesco Mainardi, Marco Raberto, Rudolf Gorenflo, Enrico Scalas, Fractional calculus and continuous-time finance II: the waiting-time distribution, Phys. A 287 (3) (2000) 468–481.
[10] M.S. Tavazoei, M. Haeri, M. Attari, S. Bolouki, M. Siami, More details on analysis of fractional-order van der Pol oscillator, J. Vib. Control 15 (6) (2009) 803–819.
[11] R.S. Barbosa, J.A.T. Machado, B. Vinagre, A. Calderon, Analysis of the Van der Pol oscillator containing derivatives of fractional order, J. Vib. Control 13 (9–10) (2007) 1291–1301.
[12] S. Jesus, J.A.T. Machado, J.B. Cunha, Fractional electrical impedances in botanical elements, J. Vib. Control 14 (2008) 1389–1402.
[13] I.S. Jesus, J.A.T. Machado, J.B. Cunha, Fractional order electrical impedance of fruits and vegetables, in: Proceedings of the 25th IASTED International

Conference MODELLING, IDENTIFICATION, AND CONTROL, February 6–8, 2006, Lanzarote, Canary Islands, Spain.

[14] J.R. King, Exact polynomial solutions to some non linear diffusion equation, Phys. D 64 (1993) 35–65.

[15] R. Cherniha, J.R. King, Nonlinear reaction diffusion systems with variable diffusitivities, Lie sysmetries, ans.tze and exact solutions, J. Math. Anal. Appl. 308 (2005) 11–35.

[16] N. Ozalp, E. Demirci, A fractional order SEIR model with vertical transmission, Math. Comput. Model. 54 (2011) 1–6.

[17] C.H. Kou, Y. Yan, J. Liu, Stability analysis for fractional differential equations and their applications in the models of HIV-1 infection, Comput. Model. Eng. Sci. 39 (2009) 301–317.

[18] J. Malmivuo, R. Plonsey, Bioelectromagnetism: Principles and Applications of Bioelectric and Biomagnetic Fields, 1st Edition., Oxford University Press, USA, 1995.

[19] M. Mophou Gisèle, Optimal control of fractional diffusion equation, Comput. Math. Appl. 61 (1) (2011) 68–78.

[20] D. René, M. Mophou Gisle, Valmorin Vincent, Optimal control of a nonhomogeneous Dirichlet boundary fractional diffusion equation, Comput. Math. Appl. 623 (2011) 1472–1481.

[21] S. Shen, F. Liu, V. Anh, I. Turner, J. Chen, A characteristic difference method for the variable-order fractional advection-diffusion equation, J. Appl. Math. Comput. 42 (1–2) (2013) 371–386.

[22] Z. Yunying, D. Weihua, W. Yujiang, A note on the finite element method for the space-fractional advection diffusion equation, Comput. Math. Appl. 59 (5) (2010) 1718–1726.

[23] Y.Z. Povstenko, Fundamental solutions to time-fractional advection diffusion equation in a case of two space variables, Math. Prob. Eng. 2014 (2014). 705364–705371.

[24] Z. Haixiang, H. Xuli, Quasi-wavelet method for time-dependent fractional partial differential equation, Int. J. Comput. Math. 90 (11) (2013) 2491–2507.

[25] E. Pindza, K.M. Owolabi, Fourier spectral methods for higher order space fractional reaction diffusion equations, Commun. Nonlinear Sci. Numer. Simul. 40 (2016) 112–128.

[26] A. Abdon, B.S. Brahim, Solving partial differential equation with space- and time-fractional derivatives via homotopy decomposition method, Math. Probl. Eng. 2013 (2013). 318590–318599.

[27] O.P. Agrawal, Solution for a fractional diffusion-wave equation defined in a bounded domain, Nonlinear Dyn. 29 (1–4) (2011) 145–155.

[28] S.S. Ray, R.K. Bera, Analytical solution of a fractional diffusion equation by Adomian decomposition method, Appl. Math. Comput. 174 (1) (2006) 329–336.

[29] A. Arikoglu, I. Ozkol, Solution of fractional differential equations by using differential transform method, Chaos Solitons Fractals 34 (2007) 1473–1481.

[30] L. Song, H. Zhang, Application of homotopy analysis method to fractional KdV–Burgers–Kuramoto equation, Phys. Lett. A 367 (2007) 88–94.

[31] Y. Khan, Q. Wu, Homotopy perturbation transform method for nonlinear equations using He's polynomials, Comput. Math. Appl. 61 (2011) 1963–1967.

[32] L. Debnath, D. Bhatta, Integral Transforms and Their Applications, Chapman & Hall/CRC Press, 2007.

[33] J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178 (1999) 257–262.